

# **Lecture 2**

## Formalising Problems

# Problems as Functions

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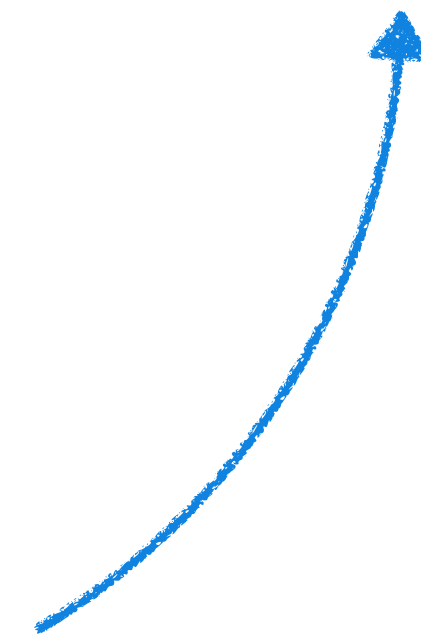
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